

AFFINE GEOMETRIC CRYSTALS IN UNIPOTENT LOOP GROUPS

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ABSTRACT. We study products of the affine geometric crystal of type A corresponding to symmetric powers of the standard representation. The quotient of this product by the R -matrix action is constructed inside the unipotent loop group. This quotient crystal has a semi-infinite limit, where the crystal structure is described in terms of limit ratios previously appearing in the study of total positivity of loop groups.

1. INTRODUCTION

Geometric crystals were invented by Berenstein and Kazhdan [1, 2] as birational analogues of Kashiwara's crystal graphs. Suppose X is a geometric crystal. Then the product X^m is also a geometric crystal, and in certain cases [4] one has a (birational) R -matrix $R : X \times X \rightarrow X \times X$ giving rise to a birational action of the symmetric group S_m on X^m . Since the R -matrix is an isomorphism of geometric crystals, the crystal structure of X^m can be considered as S_m -invariants of X^m . Our investigations began with expressing the crystal structure in terms of the invariant rational functions $\mathbb{C}(X^m)^{S_m}$ in the special case that $X = X_M$ is the *basic geometric crystal* corresponding to symmetric powers of the standard representation of $U'_q(\mathfrak{sl}_n)$. Equivalently, we construct the *quotient crystal* of X_M^m by S_m . It turns out that this quotient crystal can be constructed inside the unipotent loop group U in an extremely natural way.

The *basic geometric crystal* X_M of type $A_{n-1}^{(1)}$ is the variety

$$X_M = \{x = (x^{(1)}, x^{(2)}, \dots, x^{(n)}) \mid x^{(i)} \in \mathbb{C}^*\} = \mathbb{C}^n,$$

equipped with a distinguished collection of rational functions $\varepsilon_i(x) = x^{(i+1)}$, $\varphi_i(x) = x^{(i)}$, $\gamma_i(x) = x^{(i)}/x^{(i+1)}$ and a collection of rational \mathbb{C}^* -actions $e_i^c : X_M \rightarrow X_M$, $c \in \mathbb{C}^*$ satisfying certain relations (see Section 2.1). Our X_M is essentially the geometric crystal $\mathcal{B}_L(A_{n-1}^{(1)})$ of Kashiwara, Nakashima, and Okado [4, Section 5.2], with the dependence on L removed. It is a geometric analogue [3] of a limit of perfect (combinatorial) crystals, the latter playing an important role in the theory of vertex models.

Now consider the product $X = X_1 \times X_2 \times \dots \times X_m$, where each $X_i \simeq X_M$. In this case, the birational R -matrix $R_i : X_i \times X_{i+1} \rightarrow X_{i+1} \times X_i$ has been explicitly calculated [7]. This same rational transformation appeared in the study of total positivity in loop groups. Let $G = GL_n(\mathbb{C}((t)))$ denote the formal loop group, and let $U \subset G$ denote the maximal unipotent subgroup of G . In [5], we defined certain elements $M(x^{(1)}, \dots, x^{(n)}) \in U$ (see Section 3.2), called *whirls*, depending on n parameters $x^{(1)}, \dots, x^{(n)}$. We showed that any totally nonnegative element $g \in U_{\geq 0}$ whose matrix entries were polynomials, was a

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product of such whirls. It turns out that generically there is a unique non-trivial rational transformation of parameters $(x, y) \mapsto (u, w)$ such that

$$M(x^{(1)}, x^{(2)}, \dots, x^{(n)})M(y^{(1)}, y^{(2)}, \dots, y^{(n)}) = M(u^{(1)}, u^{(2)}, \dots, u^{(n)})M(w^{(1)}, w^{(2)}, \dots, w^{(n)}).$$

Up to a shift of indices, this transformation coincides with the birational R -matrices R_i described above.

Let $U^{\leq m} \subset U$ denote the closure of the set of elements $\{g = \prod_{i=1}^m M(x_i)\}$ which can be expressed as the product of m whirls. We show that $\mathbb{C}(U^{\leq m}) = \mathbb{C}(X_M^m)^{S_m}$ and that the geometric crystal structure of X_M^m descends to $U^{\leq m}$, where it can be described explicitly in terms of left and right multiplication by one-parameter subgroups of U (Theorems 3.1 and 4.1 and Corollary 4.4). This is reminiscent of the construction [2] of a geometric crystal from a unipotent bicrystal, though $U^{\leq m}$ is not closed under multiplication by U .

One intriguing observation we make is that the geometric crystal $U^{\leq m}$ has a limit as $m \rightarrow \infty$, which morally one may think of as a quotient of the semi-infinite product X_M^∞ . It gives rise to a “geometric crystal” structure on the whole unipotent loop group U , where ε_k, φ_k are no longer rational functions, but are asymptotic *limit ratios* of the matrix coefficients of U (Theorem 3.2). These limit ratios played a critical role in the factorization of totally nonnegative elements [5]. It points to a deeper connection which we have yet to understand.

The matrix coefficients of $U^{\leq m}$ give a natural set of (algebraically independent) generators of $\mathbb{C}(X_M^m)^{S_m}$. These matrix coefficients are the *loop elementary symmetric functions* $e_r^{(s)}(x_1, \dots, x_m)$. The invariants also contain distinguished elements $s_\lambda^{(s)}(x_1, \dots, x_m)$, called the *loop Schur functions*. It was observed in [6] that the (birational) intrinsic energy function of X_M^m can be expressed as a loop Schur function of dilated staircase shape. In particular, energy is a polynomial, unlike the rational functions ε_k, φ_k . We explicitly describe the crystal operator action e_k^c on loop Schur functions (Theorem 5.3).

2. PRODUCTS OF GEOMETRIC CRYSTALS

2.1. Geometric crystals. We shall use [4] as our main reference for affine geometric crystals. In this paper we shall consider affine geometric crystals of type A . Fix $n > 1$. Let $A = (a_{ij})_{i,j \in \mathbb{Z}/n\mathbb{Z}}$ denote the $A_{n-1}^{(1)}$ Cartan matrix. Thus if $n > 2$ then $a_{ii} = 2$, $a_{ij} = -1$ for $|i - j| = 1$, and $a_{ij} = 0$ for $|i - j| > 1$. For $n = 2$, we have $a_{11} = a_{22} = 2$ and $a_{12} = a_{21} = -2$.

Let $(X, \{e_i\}_{i \in \mathbb{Z}/n\mathbb{Z}}, \{\varepsilon_i\}_{i \in \mathbb{Z}/n\mathbb{Z}}, \{\gamma_i\}_{i \in \mathbb{Z}/n\mathbb{Z}})$ be an affine geometric crystal for $A_{n-1}^{(1)}$. Thus, X is a complex algebraic variety, $\varepsilon_i : X \rightarrow \mathbb{C}$ and $\gamma_i : X \rightarrow \mathbb{C}$ are rational functions, and $e_i : \mathbb{C}^* \times X \rightarrow X$ ($(c, x) \mapsto e_i^c(x)$) is a rational \mathbb{C}^* -action, satisfying:

- (1) The domain of $e_i^1 : X \rightarrow X$ is dense in X for any $i \in \mathbb{Z}/n\mathbb{Z}$.
- (2) $\gamma_j(e_i^c(x)) = c^{a_{ij}} \gamma_j(x)$ for any $i, j \in \mathbb{Z}/n\mathbb{Z}$.
- (3) $\varepsilon_i(e_i^c(x)) = c^{-1} \varepsilon_i(x)$.
- (4) for $i \neq j$ such that $a_{ij} = 0$ we have $e_i^c e_j^{c'} = e_j^{c'} e_i^c$.
- (5) for $i \neq j$ such that $a_{ij} = -1$ we have $e_i^c e_j^{cc'} e_i^{c'} = e_j^{c'} e_i^{cc'} e_j^c$.

We often abuse notation by just writing X for the geometric crystal. We define $\varphi_i = \gamma_i \varepsilon_i$, and sometimes define a geometric crystal by specifying φ_i and ε_i , instead of γ_i .

2.2. Products. If X, X' are affine geometric crystals, then so is $X \times X'$ [1, 4]. Let $(x, x') \in X \times X'$. Then

$$(1) \quad \varepsilon_k(x, x') = \frac{\varepsilon_k(x)\varepsilon_k(x')}{\varphi_k(x') + \varepsilon_k(x)} \quad \varphi_k(x, x') = \frac{\varphi_k(x)\varphi_k(x')}{\varepsilon_k(x) + \varphi_k(x')}$$

and

$$e_k^c(x \otimes x') = (e_k^{c^+} x, e_k^{c/c^+} x')$$

where

$$(2) \quad c^+ = \frac{c\varphi_k(x') + \varepsilon_k(x)}{\varphi_k(x') + \varepsilon_k(x)}.$$

Remark 1. Note that our notations differ from those in [4] by swapping left and right in the product, and this agrees with [6]. Note also that the birational formulae we use should be tropicalized using $(\min, +)$ (rather than $(\max, +)$) operations to yield the formulae for combinatorial crystals.

2.3. The basic geometric crystal. We now introduce a geometric crystal X_M which we call the *basic geometric crystal* of type $A_{n-1}^{(1)}$. This is a geometric analogue of a limit of perfect crystals. It is essentially the geometric crystal $\mathcal{B}_L(A_{n-1}^{(1)})$ of [4, Section 5.2].

We have $X_M = \{x = (x^{(1)}, x^{(2)}, \dots, x^{(n)}) \mid x^{(i)} \in \mathbb{C}^*\}$ and

$$\varepsilon_i(x) = x^{(i+1)} \quad \varphi_i(x) = x^{(i)} \quad \gamma_i(x) = x^{(i)}/x^{(i+1)}$$

and

$$e_i^c : (x^{(1)}, x^{(2)}, \dots, x^{(n)}) \longmapsto (x^{(1)}, \dots, cx^{(i)}, c^{-1}x^{(i+1)}, \dots, x^{(n)}).$$

That X_M is an affine geometric crystal is shown in [4].

3. GEOMETRIC CRYSTAL STRUCTURES ON THE UNIPOTENT LOOP GROUP

3.1. Unipotent loop group. Let $G = GL_n(\mathbb{C}((t)))$ denote the formal loop group, consisting of non-singular $n \times n$ matrices with complex formal Laurent series coefficients. If $g = (g_{ij})_{i,j=1}^n \in G$ where $g_{ij} = \sum_k g_{ij}^k t^k$, we let $Y = Y(g) = (y_{rs})_{r,s \in \mathbb{Z}}$ denote the infinite periodic matrix defined by $y_{i+kn, j+k'n} = g_{ij}^{k'-k}$ for $1 \leq i, j \leq n$. For the purposes of this paper we shall always think of formal loop group elements as infinite periodic matrices.

The unipotent loop group $U \subset G$ is defined as those Y which are upper triangular, with 1's along the main diagonal. We let $U^{\leq m} \subset U$ denote the subset consisting of infinite periodic matrices supported on the m diagonals above the main diagonal. Thus if $Y \in U$ then $Y \in U^{\leq m}$ if and only if $y_{ij} = 0$ for $j - i > m$. It is clear that $U^{\leq m} \simeq \mathbb{C}^{mn}$.

3.2. Whirls and Chevalley generators. For $k \in \mathbb{Z}/n\mathbb{Z}$ and $a \in \mathbb{C}$, define the Chevalley generator $u_k(a) \in U$ by

$$(u_k(a))_{ij} = \begin{cases} 1 & \text{if } j = i \\ a & \text{if } j = i + 1 = k + 1 \\ 0 & \text{otherwise.} \end{cases}$$

These are standard one-parameter subgroups of U corresponding to the simple roots.

For $(x^{(1)}, x^{(2)}, \dots, x^{(n)}) \in \mathbb{C}^n$, define the *whirl* $M(x) = M(x^{(1)}, x^{(2)}, \dots, x^{(n)})$ [5] to be the infinite periodic matrix with

$$M(x)_{i,j} = \begin{cases} 1 & \text{if } j = i; \\ x^{(i)} & \text{if } j = i + 1; \\ 0 & \text{otherwise.} \end{cases}$$

Here and elsewhere the upper indices are to be taken modulo n .

3.3. Geometric crystal structure. Let $m \in \{1, 2, \dots\}$. For $k \in \mathbb{Z}/n\mathbb{Z}$, define rational functions ε_k and φ_k on U by

$$\varepsilon_k(Y) = \frac{y_{k+1,k+m+1}}{y_{k+1,k+m}} \quad \varphi_k(Y) = \frac{y_{k,k+m}}{y_{k+1,k+m}}.$$

(These rational functions, and indeed the geometric crystal structure as well, extend to G . However, we shall only consider the unipotent loop group.)

Theorem 3.1 (Unipotent loop group geometric crystal for finite m). *Fix $m \in \{1, 2, \dots\}$. The formal loop group U , the rational functions $\varepsilon_k, \varphi_k : U \rightarrow \mathbb{C}$, and the map*

$$(3) \quad e_k^c : Y \mapsto u_k((c-1)\varphi_k) Y u_{k+m}((c^{-1}-1)\varepsilon_k)$$

form a geometric crystal. Furthermore, the subvariety $U^{\leq m}$ is invariant under e_k^c , and thus is a geometric subcrystal.

This result is not difficult to prove by direct calculation, but we omit the proof as it essentially follows from Corollary 4.4 (which shows that one has a geometric crystal structure on $U^{\leq m}$).

Remark 2. The formula (3) is essentially the same formula as that used by Berenstein and Kazhdan [2, (2.14)] to define a geometric crystal from a unipotent crystal. Indeed, U clearly has a $U \times U$ action (though $U^{\leq m}$ does not). However, despite the simplicity of our construction, we have been unable to put it inside their framework of unipotent crystals.

3.4. Asymptotic geometric crystal structure. Let $Y \in U$. Define the limit ratios [5]

$$\varepsilon_k(Y) = \lim_{m \rightarrow \infty} \frac{y_{k-m,k+1}}{y_{k-m,k}} \quad \varphi_k(Y) = \lim_{m \rightarrow \infty} \frac{y_{k,k+m}}{y_{k+1,k+m}}$$

assuming the limits exist. Since ε_k and φ_k are not rational functions, we cannot use them to construct an algebraic geometric crystal. However, treating ε_k and φ_k as functions with a restricted domain, we can still formally construct a geometric crystal using the formula (3).

Theorem 3.2 (Unipotent loop group geometric crystal for $m = \infty$). *The functions $\varepsilon_k, \varphi_k : U \rightarrow \mathbb{C}$, and the map*

$$e_k^c : Y \mapsto u_k((c-1)\varphi_k) Y u_k((c^{-1}-1)\varepsilon_k)$$

satisfy the relations (2), (3), (4), (5) of a geometric crystal on U (see Section 2.1).

Note that if ε_k and φ_k are defined at $Y \in U$, so is e_k^c , and in addition $e_k^c(Y)$ also has this property.

Proof. For simplicity we assume $n > 2$ and suppose that φ_k and ε_k are defined for $Y \in U$. We have $\varphi_j(Yu_k(a)) = \varphi_j(Y)$ and $\varepsilon_j(u_k(a)Y) = \varepsilon_j(Y)$, and

$$\varphi_j(u_k(a)Y) = \begin{cases} \varphi_k(Y) + a & \text{if } k = j \\ \varphi_{k-1}(Y)/(1 + a/\varphi_k(Y)) & \text{if } k - 1 = j \\ \varphi_j(Y) & \text{otherwise.} \end{cases}$$

$$\varepsilon_j(Yu_k(a)) = \begin{cases} \varepsilon_k(Y) + a & \text{if } k = j \\ \varepsilon_{k+1}(Y)/(1 + a/\varepsilon_k(Y)) & \text{if } k + 1 = j \\ \varepsilon_j(Y) & \text{otherwise.} \end{cases}$$

Relations (2),(3),(4) are immediate. To obtain (5), one uses the relation $u_k(a)u_{k+1}(b)u_k(c) = u_{k+1}(bc/(a+c))u_k(a+c)u_{k+1}(ab/(a+c))$ to get

$$\begin{aligned} & u_k \left((c' - 1) \frac{c\phi_k}{1 + (cc' - 1)\phi_{k+1}/\phi_{k+1}} \right) u_{k+1}((cc' - 1)\phi_{k+1})u_k((c - 1)\phi_k) \\ &= u_{k+1}((c - 1)c'\phi_{k+1})u_k((cc' - 1)\frac{\phi_k}{1 + (c' - 1)\phi_k/\phi_k})u_{k+1}((c' - 1)\phi_{k+1}) \end{aligned}$$

where $\phi_k = \phi_k(Y)$. A similar equality for ε_k gives (5).

The last statement of the theorem is straightforward. \square

In place of axiom (1) of a geometric crystal, we may note that the domain of definition of $\varepsilon_k, \varphi_k, e_k^c$ is dense in U under (matrix-)entrywise convergence. However, ε_k and φ_k are not continuous in this topology.

Remark 3. The limit ratios ε_k and φ_k were introduced in [5] for the study of the totally nonnegative part $U_{\geq 0}$ of the loop group. Indeed, these limits always exist for totally nonnegative elements which are not supported on finitely many diagonals. They point to a deeper connection between total nonnegativity and crystals.

4. PRODUCT CRYSTAL STRUCTURE AND S_m -INVARIANTS

4.1. Birational R -matrix and invariants. Let $X = X_1 \times X_2 \times \cdots \times X_m$ where each $X_i \simeq X_M$ is a basic affine geometric crystal. We shall shift the indexing of the coordinates on X_i , so that

$$X_i = \{x_i = (x_i^{(i)}, x_i^{(i+1)}, \dots, x_i^{(i+n)})\}.$$

That is, $\varphi_1(x_i) = x_i^{(i)}$, and so on.

Define

$$\kappa_r(x_j, x_{j+1}) = \sum_{s=0}^{n-1} \left(\prod_{t=1}^s x_{j+1}^{(r+t)} \prod_{t=s+1}^{n-1} x_j^{(r+t)} \right).$$

In the variables $x_i^{(r)}$, the birational R -matrix acts (see [7, Proposition 3.1]) via algebra isomorphisms s_1, s_2, \dots, s_{m-1} of the field $\mathbb{C}(X)$ of rational functions in $\{x_i^{(r)}\}$, given by

$$s_j(x_j^{(r)}) = \frac{x_{j+1}^{(r+1)} \kappa_{r+1}(x_j, x_{j+1})}{\kappa_r(x_j, x_{j+1})} \quad \text{and} \quad s_j(x_{j+1}^{(r)}) = \frac{x_j^{(r-1)} \kappa_{r-1}(x_j, x_{j+1})}{\kappa_r(x_j, x_{j+1})}$$

and $s_j(x_k^{(r)}) = x_k^{(r)}$ for $k \neq j, j+1$. The birational R -matrix acts as geometric crystal isomorphisms, and thus the crystal structure of X descends to the invariants $\mathbb{C}(X)^{S_m}$.

For a sequence $x = (x_1, \dots, x_m)$ of n -tuples of complex numbers, we define $M(x) \in U$ to be the product of whirls $M(x) = M(x_1^{(1)}, \dots, x_1^{(n)}) \cdots M(x_m^{(1)}, \dots, x_m^{(n)})$. The entries of $M(x)$ have the following description. For $r \geq 1$ and $s \in \mathbb{Z}/n\mathbb{Z}$, define the *loop elementary symmetric functions*

$$e_r^{(s)}(x_1, x_2, \dots, x_m) = \sum_{1 \leq i_1 < i_2 < \dots < i_r \leq m} x_{i_1}^{(s)} x_{i_2}^{(s+1)} \cdots x_{i_r}^{(s+r-1)}.$$

By convention we have $e_r^{(s)} = 1$ if $r = 0$, and $e_r^{(s)} = 0$ if $r < 0$. Then by [5, Lemma 7.5] we have

$$(4) \quad M(x)_{i,j} = e_{j-i}^{(i)}(x_1, \dots, x_m).$$

Furthermore, it is shown in [5, Section 6] that

$$(5) \quad M(x_1)M(x_2) = M(s_1(x_1))M(s_1(x_2)).$$

The ring $\text{LSym}_m \subset \mathbb{C}[X]$ generated by the $e_r^{(s)}$ is called the ring of (whirl) loop symmetric functions.

Theorem 4.1. *We have $\mathbb{C}(X)^{S_m} = \mathbb{C}(e_r^{(s)})$. In particular, the $e_r^{(s)}$ are algebraically independent. The map $X \rightarrow U^{\leq m}$ given by*

$$(x_1, \dots, x_m) \mapsto M(x_1)M(x_2) \cdots M(x_m)$$

identifies $\text{LSym}_m = \mathbb{C}[e_r^{(s)}]$ with the coordinate ring $\mathbb{C}[U^{\leq m}]$.

Proof. It follows from (5) that $e_r^{(s)} \in \mathbb{C}(X)^{S_m}$ (see also [7]). Now consider the map $g : X \rightarrow \mathbb{C}^{mn}$ given by $x \mapsto (e_r^{(s)})_{r \in \{1, 2, \dots, m\}}^{s \in \mathbb{Z}/n\mathbb{Z}}$. It is shown in [5, Corollary 6.4, Proposition 8.2] that when $x_i^{(k)}$ are positive real numbers and the products $R_i = \prod_{k \in \mathbb{Z}/n\mathbb{Z}} x_i^{(k)}$ are all distinct the map g is $n!$ to 1. Since this is a Zariski-dense subset of X , we conclude that $[\mathbb{C}(X) : \mathbb{C}(e_r^{(s)})] = m!$. But then we must have $\mathbb{C}(X)^{S_m} = \mathbb{C}(e_r^{(s)})$. Since the transcendence degree of $\mathbb{C}(X)^{S_m}$ is mn , it follows that the $e_r^{(s)}$ are algebraically independent. The last statement follows immediately from (4). \square

Remark 4. In a future work we shall show the stronger result that $\mathbb{C}(X)^{S_m} \cap \mathbb{C}[X] = \mathbb{C}[e_r^{(s)}]$.

4.2. Crystal structure in terms of invariants. The aim of this section is to completely describe the geometric crystal structure of X in terms of the invariants $e_r^{(s)}$.

Theorem 4.2. *Let $x = (x_1, x_2, \dots, x_m) \in X$. We have*

$$\varepsilon_k(x) = \frac{e_m^{(k+1)}}{e_{m-1}^{(k+1)}} \quad \varphi_k(x) = \frac{e_m^{(k)}}{e_{m-1}^{(k+1)}} \quad \gamma_k(x) = \frac{e_m^{(k)}}{e_m^{(k+1)}}.$$

Proof. We proceed by induction on m . For $m = 1$, we have $\varepsilon_k(x) = x_1^{(k+1)}$ and $\varphi_k(x) = x_1^{(k)}$, agreeing with the theorem.

Let $x' = (x_1, \dots, x_{m-1}) \in X_1 \times X_2 \times \dots \times X_{m-1}$. Supposing the result is true for $m-1$, we calculate that

$$\begin{aligned} \varepsilon_k(x', x_{m-1}) &= \frac{\varepsilon_k(x') \varepsilon_k(x_m)}{\varphi_k(x_m) + \varepsilon_k(x')} \\ &= \frac{e_{m-1}^{(k+1)}(x_1, \dots, x_{m-1}) x_m^{(k+m)}}{e_{m-2}^{(k+1)}(x_1, \dots, x_{m-1}) x_m^{(k+m-1)} + e_{m-1}^{(k+1)}(x_1, \dots, x_{m-1})} \\ &= \frac{e_m^{(k+1)}(x_1, \dots, x_m)}{e_{m-1}^{(k+1)}(x_1, \dots, x_m)}. \end{aligned}$$

The calculation for φ_k is similar. For γ_k we use $\gamma_k = \varphi_k / \varepsilon_k$. \square

The next theorem completes the description of the geometric crystal structure in terms of the $e_r^{(s)}$.

Theorem 4.3. *The map $(e_k^c)^* : \mathbb{C}(X)^{S_m} \rightarrow \mathbb{C}(X)^{S_m}$ induced by $e_k^c : X \rightarrow X$ is given by*

$$M(x) \mapsto u_k((c-1)\varphi_k) M(x) u_{k+m}((c^{-1}-1)\varepsilon_k).$$

In other words $e_r^{(s)}$ is sent to

- (1) $e_r^{(k)} + (c-1)\varphi_k e_{r-1}^{(k+1)}$ if $s = k \neq k+m-r+1 \pmod n$
- (2) $e_r^{(k+m-r+1)} + (c^{-1}-1)\varepsilon_k e_{r-1}^{(k+m-r+1)}$ if $s = k+m-r+1 \neq k \pmod n$
- (3) $e_r^{(k)} + (c-1)\varphi_k e_{r-1}^{(k+1)} + (c^{-1}-1)\varepsilon_k e_{r-1}^{(k)} - \frac{(1-c)^2}{c} \varepsilon_k \varphi_k e_{r-2}^{(k+1)}$ if $s = k+m-r+1 = k \pmod n$
- (4) $e_r^{(s)}$ otherwise.

Example 1. Let $m = n = 2$. The action of e_1^c on the crystal is induced by the transformation $M(x) \mapsto$

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \dots & 1 & (c-1)\varphi_1 & 0 & 0 & \dots \\ \dots & 0 & 1 & 0 & 0 & \dots \\ \dots & 0 & 0 & 1 & (c-1)\varphi_1 & \dots \\ \dots & 0 & 0 & 0 & 1 & \dots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} M(x) \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \dots & 1 & (1-c)\varepsilon_1/c & 0 & 0 & \dots \\ \dots & 0 & 1 & 0 & 0 & \dots \\ \dots & 0 & 0 & 1 & (1-c)\varepsilon_1/c & \dots \\ \dots & 0 & 0 & 0 & 1 & \dots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where

$$M(x) = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \dots & 1 & e_1^{(1)} & e_2^{(1)} & 0 & \dots \\ \dots & 0 & 1 & e_1^{(2)} & e_2^{(2)} & \dots \\ \dots & 0 & 0 & 1 & e_1^{(1)} & \dots \\ \dots & 0 & 0 & 0 & 1 & \dots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \varphi_1 = e_2^{(1)}/e_1^{(2)}, \quad \varepsilon_1 = e_2^{(2)}/e_1^{(2)}.$$

Proof. The proof is by induction on m . For $m = 1$ the statement is easily verified from the definition of basic geometric crystal. Assume now $m > 1$. We let $x = (x', x_m)$, where $x' = (x_1, \dots, x_{m-1})$. Then

$$\begin{aligned} M(x') &\mapsto u_k((c^+ - 1)\varphi_k(x')) M(x') u_{k+m-1}((c^+ - 1)\varepsilon_k(x')), \\ M(x_m) &\mapsto u_{k+m-1}((c/c^+ - 1)\varphi_k(x_m)) M(x_m) u_{k+m}((c^+/c - 1)\varepsilon_k(x_m)). \end{aligned}$$

Plugging in

$$c^+ = \frac{c\varphi_k(x_m) + \varepsilon_k(x')}{\varphi_k(x_m) + \varepsilon_k(x')}$$

and using (1) we obtain

$$\begin{aligned} M(x')M(x_m) &\mapsto u_k\left(\frac{(c-1)\varphi_k(x_m)}{\varphi_k(x_m) + \varepsilon_k(x')}\varphi_k(x')\right) M(x') u_{k+m-1}\left(\frac{(1-c)\varphi_k(x_m)}{c\varphi_k(x_m) + \varepsilon_k(x')}\varepsilon_k(x')\right) \\ &u_{k+m-1}\left(\frac{(c-1)\varepsilon_k(x')}{c\varphi_k(x_m) + \varepsilon_k(x')}\varphi_k(x_m)\right) M(x_m) u_{k+m}\left(\frac{(1-c)\varepsilon_k(x')}{c(\varphi_k(x_m) + \varepsilon_k(x'))}\varepsilon_k(x_m)\right) = \\ &u_k((c-1)\varphi_k(x)) M(x) u_{k+m}((c^{-1} - 1)\varepsilon_k(x)). \end{aligned}$$

□

If X is a geometric crystal and Γ is a group of crystal automorphisms of X , we say that a geometric crystal X' is the quotient of X by Γ , if we have a rational map $X \rightarrow Y$, commuting with the geometric crystal structure, and inducing $\mathbb{C}(Y) \simeq \mathbb{C}(X)^\Gamma$.

Corollary 4.4. *The map $(x_1, \dots, x_m) \mapsto M(x_1) \cdots M(x_m)$ identifies the quotient of the product geometric crystal $X = X_M^m$ by the S_m birational R -matrix action with the geometric crystal on $U^{\leq m}$ of Theorem 3.1.*

Remark 5. Our whirls play a similar role to the M -matrices of [4], which have been studied for all affine types. This suggests that Theorem 4.3 and also our work on total positivity [5] may have a generalization to other types in the same spirit.

5. ACTION OF CRYSTAL OPERATORS ON THE ENERGY FUNCTION

We recall the definition of the *loop Schur functions* that were introduced in [5]. Let λ/μ be a skew Young diagram shape. A square $s = (i, j)$ in the i -th row and j -th column has *content* $c(s) = i - j$. We caution that our notion of content is the negative of the usual one. Recall that a semistandard Young tableaux T with shape λ/μ is a filling of each square $s \in \lambda/\mu$ with an integer $T(s) \in \mathbb{Z}_{>0}$ so that the rows are weakly-increasing, and columns are increasing. For $r \in \mathbb{Z}/n\mathbb{Z}$, the r -weight x^T of a tableaux T is given by $x^T = \prod_{s \in \lambda/\mu} x_{T(s)}^{(c(s)+r)}$.

We shall draw our shapes and tableaux in English notation:

$$\begin{array}{ccccccc} \times & \times & \times & \times & \times & \circ & \square & \square & \bullet \\ \times & \times & \times & \circ & \square & \square & \square & \bullet & \\ \times & \times & \times & \square & \square & \bullet & & & \end{array} \quad \begin{array}{ccccccc} & & & 1 & 1 & 1 & 3 \\ 1 & 2 & 2 & 3 & 4 & & \\ 3 & 3 & 4 & & & & \end{array}$$

For $n = 3$ the 0-weight of the above tableau is $(x_1^{(1)})^2(x_3^{(1)})^3x_1^{(2)}x_2^{(2)}x_3^{(2)}x_1^{(3)}x_2^{(3)}(x_4^{(3)})^2$. We define the *loop (skew) Schur function* by

$$s_{\lambda/\mu}^{(r)}(x) = \sum_T x^T$$

where the summation is over all semistandard Young tableaux of (skew) shape λ/μ . Loop Schur functions are of significance in the theory of geometric crystals because of the following result. Let $\delta_m = (m, m-1, \dots, 1)$ denote the staircase shape of size m .

Theorem 5.1. [6, Theorem 1.2] *The birational energy function \overline{D}_B of $X = X_1 \times \dots \times X_m$ is the loop Schur function $s_{(n-1)\delta_{m-1}}^{(0)}(x_1, \dots, x_m)$.*

Note that unlike ε_k and φ_k , the birational energy function is a polynomial. We have the following analog of the Jacobi-Trudi formula.

Theorem 5.2. [5, Theorem 7.6] *We have $s_{\lambda/\mu}^{(r)} = \det(e_{\lambda'_i - \mu'_j - i + j}^{(r-j+1+\mu'_j)})$.*

This means that the skew loop Schur functions $s_{\lambda/\mu}^{(r)}(x_1, x_2, \dots, x_m)$ are minors of the infinite periodic matrix $M(x) = M(x_1) \cdots M(x_m)$.

For a skew shape λ/μ we distinguish its NW corners and SE corners (in English convention). The figure above shows the NW corners marked with \circ and the SE corners marked with \bullet . Let $A(\lambda/\mu)_k^{(r)}$ (resp. $B(\lambda/\mu)_k^{(r)}$) denote the set of NW (resp. SE) corners of λ/μ of color k , that is corner cells (i, j) with $i - j + r = k \pmod n$. The proof of the following result follows from an examination of the effect of row and column operations on Jacobi-Trudi matrices, and is omitted.

Theorem 5.3. *We have*

$$(e_k^c)^*(s_{\lambda/\mu}^{(r)}) = \sum_{\substack{B \subset B(\lambda/\mu)_{k+m}^{(r)} \\ A \subset A(\lambda/\mu)_k^{(r)} \\ A \cap B = \emptyset}} ((c^{-1} - 1)\varepsilon_k)^{|B|} ((c - 1)\varphi_k)^{|A|} s_{\lambda/\mu - A - B}^{(r)}.$$

We recover the following property of the birational energy function \overline{D}_B , parallel to the analogous property of the combinatorial energy function.

Corollary 5.4. *For $k \neq 0 \pmod n$ we have $(e_k^c)^*(\overline{D}_B) = \overline{D}_B$.*

Proof. The shape $(n-1)\delta_{m-1}$ has a single NW corner of color 0 and all its SE corners are of color m . Thus only for $k = 0$ the sets $A((n-1)\delta_{m-1})_k^{(0)}$ and $B((n-1)\delta_{m-1})_{k+m}^{(0)}$ are non-empty. \square

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